

Linear Algebra I

22/01/2019, Tuesday, 14:00 – 17:00

You are **NOT** allowed to use any type of calculators.

1 Systems of linear equations

(1 + 4 + 3 + (1 + 3 + 3) = 15 pts)

Consider the following system of linear equations in the unknowns v , w , x , y , and z where α is a real number:

$$\begin{aligned} 2v + 2w + 2x + 4y &= 2 \\ & 2w + 4x + 2y + 2z = 0 \\ 4v + w + 5y + z &= 4 \\ 6v + 3w + 2x + 9y + z &= \alpha \end{aligned}$$

- Write down the augmented matrix.
- By performing elementary row operations, put the augmented matrix into row echelon form.
- Determine all values of α so that the system is consistent.
- For the values of α found above,
 - determine the *lead* and *free* variables.
 - put the augmented matrix into *reduced* row echelon form by performing elementary row operations.
 - find the solution set.

2 Determinants

(9 + 6 = 15 pts)

Consider the matrix

$$M = \begin{bmatrix} \alpha & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

where α is real number.

- By using *only* row/column operations, find the determinant of M .
- Determine all values of α such that M is nonsingular.

3 Least squares problem

(15 pts)

Find the line of the form $y = a + bx$ that gives the best least squares approximation to the points:

$$\begin{array}{c|c|c|c} x & 1 & 1 & 0 \\ y & 0 & 1 & 1 \end{array}$$

4 Vector spaces $((3 + 4) + (4 + 4) = 15 \text{ pts})$

(a) Let J be the 3×3 matrix given by

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

- (i) Is the set $S = \{A \in \mathbb{R}^{3 \times 3} \mid AJ = JA\}$ a subspace?
(ii) If it is so, find a basis for S and determine its dimension.

(b) Let M be a 2×2 matrix.

- (i) Let $L_M : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^{2 \times 2}$ given by $L_M(X) = MX + XM$. Show that L_M is a linear transformation.
(ii) Take $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Write down the matrix representation of L_M using the following basis for $\mathbb{R}^{2 \times 2}$:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

5 Characteristic polynomial, determinant, and trace $(3 + 4 + 4 + 4 = 15 \text{ pts})$

Let M be the 3×3 matrix given by

$$M = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}$$

where a , b , and c are real numbers.

- (a) By using the relationship between the determinant and eigenvalues of a matrix, show that 0 is an eigenvalue of M .
(b) By using the definition of eigenvalue, show that $a + b + c$ is an eigenvalue of M .
(c) By using the relationship between the trace and eigenvalues of a matrix, show that the characteristic polynomial of M is given by $p_M(\lambda) = \lambda^2(\lambda - a - b - c)$.
(d) Suppose that $a + b + c \neq 0$. Show that M is diagonalizable.

6 Eigenvalues/vectors $(4 + 5 + 2 + 4 = 15 \text{ pts})$

Consider the matrix

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

- (a) Find the eigenvalues of M .
(b) Show that M is diagonalizable.
(c) Find a matrix X that diagonalizes M .
(d) Find e^M by using the matrix X .

10 pts free